UDC 62-50

UNBIASED COORDINATE TRANSFORMATION

PMM Vol. 40, № 4, 1976, pp. 736-739
L. E. SHIROKOV
(Moscow)
(Received April 29, 1975)

We have obtained formulas for the unbiased transition from one coordinate system to another, when the angles defining the relative positions of the coordinate systems are measured with errors. The cases of symmetric and asymmertic distributions of measurement errors are considered. The transition from one coordinate system to another in the trajectory problems of flight dynamics is effected by formulas (see [1], for example) which assume exact knowledge of the angles defining the relative position of the coordinate systems. Since these angles are usually measured with errors, the coordinate transformation is carried out erroneously, and the transition formulas give a bias to the coordinates. This bias is undesirable for two reasons: firstly, it leads to systematic errors which are substantial when solving a number of problems (navigational problems, for instance), and secondly, they are practically unfilterable. However, it is possible to construct transition formulas which ensure an unbiased coordinate transformation.

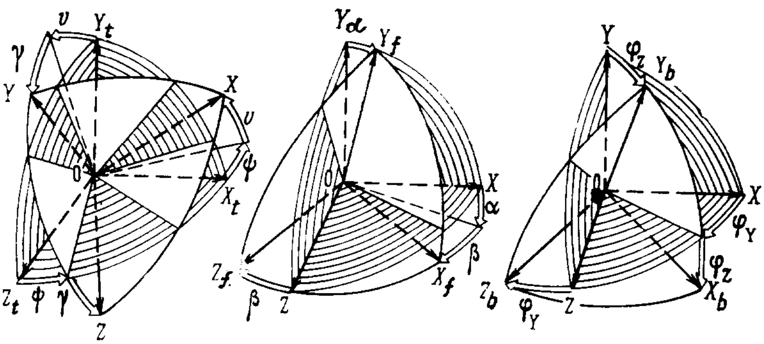

Fig. 1

In this paper we give a method for obtaining these formulas under a symmetric and an asymmetric distribution of angle measurement errors, and as an example we compute the matrices of unbiased transition for the following pairs of coordinate systems (Fig. 1): terrestrial $OX_tY_tZ_t$ -- body-**axes** $OXYZ$, body-axes $OXYZ$ — flow-axes $OX_fY_fZ_f$, body-axes $OXYZ$ — beam-axes $OX_bY_bZ_b$. The essence of the method is the preliminary computation of the magnitude of the bias and, on the basis of this, the determination of corrections to the expressions for the direction cosines, such that the bias becomes equal to zero.

1. Unbiased coordinate transformation under a symmetric angle measurement error distribution. First of all we note that the direction cosines for the transition from one coordinate system to another are polynomials $P_{\mu\nu}$ (μ, $\nu = 1, 2, 3$) of some degree n ($n \geqslant 1$) of functions of the sines and cosines of the angles φ_i° $i = 1, \ldots, p$) defining the mutual position of the coordinate systems

$$P_{\mu\nu} = P_{\mu\nu}(\sin\varphi_1^\circ, \cos\varphi_1^\circ, \ldots, \sin\varphi_p^\circ, \cos\varphi_p^\circ)$$

Let the angles φ_i° be measured with errors ξ_i

$$\varphi_i = \varphi_i^\circ + \xi_i, \quad i = 1, \ldots, p$$

and let the distribution F_ξ of the error vector $\xi = (\xi_1, \ldots, \xi_p)$ be symmetric (for example, normal) with the parameters $\langle \xi_i \rangle = 0$ and $\langle \xi_i \xi_j \rangle = \sigma_{ij}$. The problem is to find a function $a_{\mu\nu}(\varphi_1, \ldots, \varphi_p)$ of the measured values φ_i of the angles φ_i°, such that the condition

$$\langle a_{\mu\nu}(\varphi_1, \ldots, \varphi_p) \rangle = P_{\mu\nu}$$

is satisfied.

Using the trigonometric formulas for the sum and the difference of functions, we can transform the polynomial $P_{\mu\nu}(\sin\varphi_1^\circ, \cos\varphi_1^\circ, \ldots, \sin\varphi_p^\circ, \cos\varphi_p^\circ)$ into a homogeneous first-degree polynomial of the functions sine and cosine of the linearly transformed arguments. The mean of a sum equals the sum of the means and a constant factor is taken outside of the sign for the mean; therefore, it is sufficient to obtain the problem's solution for the two elementary polynomials: $P_s = \sin m^\circ$ and $P_c = \cos m^\circ$, where m° is some linear combination of the angles: $m^\circ = a_1\varphi_1^\circ + \ldots + a_p\varphi_p^\circ$, $a_i = \pm 1$.

Let us compute $\langle \sin m \rangle$ and $\langle \cos m \rangle$, where $m = a_1\varphi_1 + \ldots + a_p\varphi_p$, under the assumption that the semi-invariants of distribution F_ξ of higher than second order equal zero (this assumption is satisfied for a normal distribution). Denoting $\Delta = m - m^\circ$, we have

$$\langle \sin m \rangle = \langle \sin(\Delta + m^\circ) \rangle = \cos m^\circ \langle \sin\Delta \rangle + \sin m^\circ \langle \cos\Delta \rangle$$

$$\langle \cos m \rangle = \langle \cos(\Delta + m^\circ) \rangle = \cos m^\circ \langle \cos\Delta \rangle - \sin m^\circ \langle \sin\Delta \rangle$$

We compute $\langle \sin\Delta \rangle$ and $\langle \cos\Delta \rangle$ by the method proposed in the Appendix in [2]

$$\langle \sin\Delta \rangle = \langle \Delta \rangle - \frac{1}{3!}\langle \Delta^3 \rangle + \ldots + (-1)^n \frac{1}{(2n+1)!}\langle \Delta^{2n+1} \rangle + \ldots = 0$$

$$\langle \cos\Delta \rangle = 1 - \frac{\sigma^2}{2} + \frac{1}{2!}\left(\frac{\sigma^2}{2}\right)^2 - \ldots + (-1)^n \frac{1}{(2n-1)!}\left(\frac{\sigma^2}{2}\right)^n + \ldots = \exp\left(-\frac{\sigma^2}{2}\right)$$

$$\sigma^2 = \langle \Delta^2 \rangle = \sum_{i=1}^{p}\sum_{j=1}^{p} a_i a_j \sigma_{ij}$$

Consequently,

$$\langle \sin m \rangle = \exp\left(-\frac{\sigma^2}{2}\right)\sin m^\circ, \quad \langle \cos m \rangle = \exp\left(-\frac{\sigma^2}{2}\right)\cos m^\circ \qquad (1.1)$$

Hence we see that we need to take

$$a_s = \exp\left(\frac{\sigma^2}{2}\right)\sin m, \quad a_c = \exp\left(\frac{\sigma^2}{2}\right)\cos m \qquad (1.2)$$

as the unknown functions. Indeed, using (1.1), we find

$$\left\langle \exp\left(\frac{\sigma^2}{2}\right)\sin m \right\rangle = \sin m^\circ, \quad \left\langle \exp\left(\frac{\sigma^2}{2}\right)\cos m \right\rangle = \cos m^\circ$$

If the angle measurement errors are independent, then $\sigma^2 = \sigma_1^2 + \ldots + \sigma_p^2$ ($\sigma_i^2 \overset{\Delta}{=} \sigma_{ii}$) and the inverse transformation of the expressions obtained is possible. As a result we find that each element of the matrix of unbiased transition differs from the corresponding direction cosine by a factor of the form $\exp(\sigma^2/2)$.

Example 1. Let us find the element a_{11} of the matrix of unbiased transition from the body-axes coordinate system to the terrestrial. The direction cosine is

$$P_{11} = \cos\psi\cos\nu = {}^1/_2\cos(\psi+\nu) + {}^1/_2\cos(\psi-\nu)$$

Further, by formulas (1. 2)

$$a_{11}^{\pm} = \exp\left(\frac{\sigma_{\psi\psi} \pm 2\sigma_{\psi\nu} + \sigma_{\nu\nu}}{2}\right)\cos(\psi\pm\nu)$$

Consequently

$$a_{11} = a_{11}^{+} + a_{11}^{-} = \frac{1}{2}\exp\left(\frac{\sigma_{\psi\psi} + 2\sigma_{\psi\nu} + \sigma_{\nu\nu}}{2}\right)\cos(\psi+\nu) +$$
$$\frac{1}{2}\exp\left(\frac{\sigma_{\psi\psi} - 2\sigma_{\psi\nu} + \sigma_{\nu\nu}}{2}\right)\cos(\psi-\nu)$$

If the measurement errors for ψ and ν are independent, then $\sigma_{\psi\nu} = 0$ and

$$a_{11} = \exp\left(\frac{\sigma_\psi^2 + \sigma_\nu^2}{2}\right)\cos\psi\cos\nu$$

The elements of the matrices of unbiased transition for the coordinate systems mentioned in the introduction are computed similarly:

terrestrial axes $OX_t Y_t Z_t$ — body-axes $OXYZ$

$$\left\| \begin{matrix} c_\psi c_\nu \cos\psi\cos\nu & (-c_\nu\cos\psi\sin\nu\cos\gamma + \sin\psi\sin\gamma)\,c_\psi c_\gamma & (c_\nu\cos\psi\sin\nu\sin\gamma + \sin\psi\times\cos\gamma)\,c_\psi c_\gamma \\ c_\nu\sin\nu & c_\nu c_\gamma\cos\nu\cos\gamma & -c_\nu c_\gamma\cos\nu\sin\gamma \\ -c_\psi c_\nu\sin\psi\cos\nu & (c_\nu\sin\psi\sin\nu\cos\gamma + \cos\psi\sin\gamma)\,c_\psi c_\gamma & (-c_\nu\sin\psi\sin\nu\sin\gamma + \cos\psi\cos\gamma)\,c_\psi c_\gamma \end{matrix} \right\|$$

body-axes $OXYZ$ — flow-axes $OX_f Y_f Z_f$

$$\left\| \begin{matrix} c_\alpha c_\beta\cos\alpha\cos\beta & c_\alpha\sin\alpha & -c_\alpha c_\beta\cos\alpha\sin\beta \\ -c_\alpha c_\beta\sin\alpha\cos\beta & c_\alpha\cos\alpha & c_\alpha c_\beta\sin\alpha\sin\beta \\ c_\beta\sin\beta & 0 & c_\beta\cos\beta \end{matrix} \right\|$$

body-axes $OXYZ$ — beam-axes $OX_b Y_b Z_b$

$$\left\| \begin{matrix} c_y c_z\cos\varphi_y\cos\varphi_z & c_y c_z\cos\varphi_y\sin\varphi_z & -c_y\sin\varphi_y \\ -c_z\sin\varphi_z & c_z\cos\varphi_z & 0 \\ c_y c_z\sin\varphi_y\cos\varphi_z & c_y c_z\sin\varphi_y\sin\varphi_z & c_y\cos\varphi_y \end{matrix} \right\|$$

Here

$$c_\lambda = \exp\frac{\sigma_\lambda^2}{2} \quad (\lambda = \psi, \nu, \gamma, \alpha, \beta), \qquad c_y = \exp\frac{\sigma_{\varphi_y}^2}{2}, \qquad c_z = \exp\frac{\sigma_{\varphi_z}^2}{2}$$

2. Unbiased coordinate transformation under an asymmetric angle measurement error distribution. Let the distribution F_ξ of the error vector $\xi = (\xi_1, \ldots, \xi_p)$ be asymmetric with the parameters $\langle\xi_i\rangle = 0$, $\langle\xi_i\xi_j\rangle = \sigma_{ij}$, $\langle\xi_i\xi_j\xi_k\rangle = s_{ijk}$. In other respects the problem's statement remains as before. The solving differs only in that now the means of $\langle\sin\Delta\rangle$ and $\langle\cos\Delta\rangle$ are computed under the assumption that the semi-invariants of the distribution F_ξ of higher than third order equal zero

$$\langle\sin\Delta\rangle = 0 - \frac{s}{3!} + \frac{s}{3!}\frac{\sigma^2}{2} - \ldots + (-1)^n\frac{s}{3!}\frac{1}{(n-1)!}\left(\frac{\sigma^2}{2}\right)^{n-1} + \ldots$$
$$\frac{1}{3!}\left(\frac{s}{3!}\right)^3 - \frac{1}{3!}\left(\frac{s}{3!}\right)^3\frac{\sigma^2}{2} + \ldots + (-1)^n\frac{1}{3!}\left(\frac{s}{3!}\right)^3\frac{1}{(n-4)!}\left(\frac{\sigma^2}{2}\right)^{n-4} + \ldots$$

$$= -\frac{s}{3!}\exp\left(-\frac{\sigma^2}{2}\right) + \frac{1}{3!}\left(\frac{s}{3!}\right)^3 \exp\left(-\frac{\sigma^2}{2}\right) - \ldots =$$

$$-\exp\left(-\frac{\sigma^2}{2}\right)\sin\frac{s}{6}, \qquad s = \langle \Delta^3 \rangle = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{k=1}^{p} a_i a_j a_k s_{ijk}$$

$$\langle \cos\Delta \rangle = 1 - \frac{\sigma^2}{2} + \ldots + (-1)^n \frac{1}{n!}\left(\frac{\sigma^2}{2}\right)^n + \ldots - \frac{1}{2!}\left(\frac{s}{3!}\right)^2 + \ldots$$

$$+ (-1)^n \frac{1}{2!}\left(\frac{s}{3!}\right)^2 \frac{1}{(n-3)!}\left(\frac{\sigma^2}{2}\right)^{n-3} + \ldots = \exp\left(-\frac{\sigma^2}{2}\right)\cos\frac{s}{6}$$

Consequently

$$\langle \sin m \rangle = \exp\left(-\frac{\sigma^2}{2}\right)\sin\left(m^\circ - \frac{s}{6}\right) \tag{2.1}$$

$$\langle \cos m \rangle = \exp\left(-\frac{\sigma^2}{2}\right)\cos\left(m^\circ - \frac{s}{6}\right)$$

and we need to take

$$a_s = \exp\left(\frac{\sigma^2}{2}\right)\sin\left(m + \frac{s}{6}\right), \qquad a_c = \exp\left(\frac{\sigma^2}{2}\right)\cos\left(m + \frac{s}{6}\right) \tag{2.2}$$

as the unknown functions. If the angle measurement errors are independent, then

$$\sigma^2 = \sum_{i=4}^{p} \sigma_i^2, \qquad s = \sum_{i=1}^{p} a_i s_i \qquad (s_i \overset{\Delta}{=} s_{iii})$$

and each element of the matrix of unbiased transition differs from the corresponding direction cosine by a factor of form $\exp(\sigma^2 / 2)$ and by an addition to the arguments of the functions sine and cosine of terms of the form $s / 6$.

Example 2. Once again let

$$P_{11} = \cos\psi^\circ \cos\nu^\circ = {}^1/_2 \cos(\psi^\circ + \nu^\circ) + {}^1/_2 \cos(\psi^\circ - \nu^\circ)$$

By formulas (2.2) we find

$$a_{11}^{\pm} = \exp\left(\frac{\sigma_{\psi\psi} \pm 2\sigma_{\psi\nu} + \sigma_{\nu\nu}}{2}\right)\cos\left(\psi \pm \nu + \frac{s_{\psi\psi\psi} \pm s_{\psi\psi\nu} + s_{\psi\nu\nu} \pm s_{\nu\nu\nu}}{6}\right)$$

As in Example 1, $a_{11} = a^+_{11} + a^-_{11}$. With independent measurement errors for ψ and ν we have $s_{\psi\psi\nu} = s_{\psi\nu\nu} = 0$ and the expression for a_{11} simplifies considerably

$$a_{11} = \exp\left(\frac{\sigma_\psi^2 + \sigma_\nu^2}{2}\right)\cos\left(\psi + \frac{s_\psi}{6}\right)\cos\left(\nu + \frac{s_\nu}{6}\right)$$

REFERENCES

1. Ostoslavskii, I. V. and Strazheva, I. V., Flight Dynamics. Aircraft Trajectories. Moscow, Mashinostroenie, 1969.
2. Shirokov, L. E., Application of the theory of conditional Markov processes to the solving of one class of filtering problems. Avtomat. i Telemekhan., № 12, 1968.

Translated by N. H. C.